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MEAN SQUARE STABILITY OF LINEAR SYSTEMS UNDER THE ACTION OF A MARKOV CHAIN

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The problem of the mean square stability of a linear system which is under the action of a Markov chain is reduced to the investigation of the stability of the system for the second moments from the solutions of the original system. The system for the second moments possesses the property that its solutions, corresponding in a specific sense to positive initial data, are positive. This property permits us to apply to the investigation of the stability problem the very well developed theory of positive operators in a linear space with a cone.

1. Equations for second moments. We consider a system of n linear differential equations

$$dx/dt = A(u)x \tag{1.1}$$

which is under the action of a homogeneous Markov chain $\{u(t),\ 0 \leqslant t < \infty\}$ with a finite number of states [1, 2]. The behavior of the Markov chain is described by the transition probabilities $p_{ij}(t) = P(t, u_i, \{u_j\})$; here the matrix $P(t) = \{p_{ij}(t)\}$ satisfies the equality $P(t) = e^{Qt}$, where Q is an infinitesimal matrix with elements

$$q_{ij} = \begin{cases} \lim\limits_{t\to 0} t^{-1} p_{ij}(t), & j \neq i \\ \lim\limits_{t\to 0} t^{-1}(p_{ii}(t) - 1), & j = i \end{cases}$$

We introduce the numbers $q_i = -q_{ii}$ $(i = 1, \ldots, N)$ and the matrices $A_k = A(u_k)$ $(k = 1, \ldots, N)$. The Markov process generated by system (1.1) is denoted, as in [2], by $\{x(t), u(t),\ 0 \leqslant t < \infty\}$. The solution of system (1.1), corresponding to the initial data $x(0) = x^\circ$, $u(0) = u_k$, is written in the form $x(t;\ x^\circ,\ u_k)$. By the norm of a vector x we mean its Euclidean norm $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$

Definition (see [1]). The trivial solution of system (1.1) is said to be asymptotically mean square stable if for any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that any solution of system (1.1) with initial conditions satisfying the inequality $\| x^\circ \| \leqslant \delta$ satisfies the inequality

$$M(\| x(t; x^\circ, u_k\|^2) < \varepsilon$$

for all $t > 0$ and all u_k and, furthermore, the mean $M(\| x(t; x^\circ, u_k) \|^2)$ tends to zero as $t \to \infty$.

In [3] it was shown that the investigation of the mean square stability of a system of stochastic differential equations can be reduced to an examination of the stability of a certain deterministic system of linear differential equations with constant coefficients. For a scalar Eq. (1.1) a system of linear differential equations with constant coefficients was described in [2], which is satisfied by the second moments and the asymptotic stability of whose trivial solution is equivalent to the mean square stability of the trivial solution of Eq. (1.1). A similar system can be obtained also in the n-dimensional case. Below we use the notation adopted in [2].

Let $\varphi(H, \{u_j\})$ be the initial distribution of the Markov process $\{x(t), u(t), 0 \leqslant t < \infty\}$. We introduce the functions

$$m_{ij}^k(t) = (f_{ij}^k(x, u_s), U_t\varphi(H, \{u_s\})) = (T_t f_{ij}^k(x, u_s), \varphi(H, \{u_s\}))$$

$$(k = 1, \ldots, N; \quad i = 1, \ldots n; \quad j = 1, \ldots, n) \tag{1.2}$$

$$f_{ij}^r(x, u_s) = \begin{cases} x_i x_j, & s = r \\ 0, & s \neq r \end{cases}$$

The functions $m_{ij}^k(t)$ are the means of the functions f_{ij}^k at the instant t under the stated initial probability distribution of the Markov process. According to Theorem 4 of [2] we have

$$\frac{dm_{ij}^r(t)}{dt} = \Big(\sum_{l=1}^n a_{il}(u_r) f_{jl}^r(x, u_s) + \sum_{l=1}^n a_{jl}(u_r) f_{il}^r(x, u_s) -$$

$$- q_s f_{ij}^r(x, u_s) + \sum_{p \neq s} q_{sp} f_{ij}^r(x, u_p),\ U_t\varphi(H, \{u_s\})\Big) \tag{1.3}$$

Taking the relation

$$\sum_{p \neq s} q_{sp} f_{ij}^r(x, u_p) = \begin{cases} q_{kr} x_i x_j, & s = k \neq r \\ 0, & s = r \end{cases} = \sum_{k \neq r} q_{kr} f_{ij}^k(x, u_s)$$

into account, from (1.3) we obtain

$$\frac{dm_{ij}^r(t)}{dt} = \sum_{l=1}^n a_{il}(u_r) m_{jl}^r + \sum_{l=1}^n a_{jl}(u_r) m_{il}^r - q_r m_{ij}^r + \sum_{k \neq r} q_{kr} m_{ij}^k \tag{1.4}$$

System (1.4) is a system of linear differential equations with constant coefficients, whose order is $\frac{1}{2}n(n+1)N$ since $m_{ij}^r = m_{ji}^r$. Let us reduce the system obtained to a more convenient form. For this we first introduce the symmetric matrices $M_r(t) = \{m_{ij}^r(t)\}$ $(r = 1, \ldots, N)$. Then system (1.4) becomes

$$\frac{dM_r}{dt} = A_r M_r + M_r A_r^* - q_r M_r + \sum_{k \neq r} q_{kr} M_k \tag{1.5}$$

We introduce further the Liapunov operators L_r acting in the space $\mathbf{M}^1$ of symmetric nth-order matrices

$$L_r M_r = A_r M_r + M_r A_r^* - q_r M_r = \left(A_r - \frac{q_r}{2}\right) M_r + M_r \left(A_r - \frac{q_r}{2}\right)^* \tag{1.6}$$

the vector $M = (M_1, \ldots, M_N)$ whose components are the symmetric matrices $M_1, \ldots, M_N$, and the operators L and Q which act in the space $\mathbf{M}^N$ of the vectors M and are given by means of the Nth-order matrices

$$\mathbf{L} = \begin{Vmatrix} L_1 & 0 & \ldots & 0 \\ 0 & L_2 & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & L_N \end{Vmatrix}, \qquad \mathbf{Q} = \begin{Vmatrix} 0 & q_{21} & \ldots & q_{N1} \\ q_{12} & 0 & \ldots & q_{N2} \\ . & . & . & . \\ q_{1N} & q_{2N} & \ldots & 0 \end{Vmatrix}$$

$$\mathbf{L}M = (L_1M_1, \ldots, L_NM_N)$$

$$\mathbf{Q}M = \Big(\sum_{k\neq 1} q_{k1}M_k, \ldots, \sum_{k\neq N} q_{kN}M_k\Big)$$

In the new notation system (1.4) is then rewritten as

$$dM/dt = \mathbf{A}M \qquad (\mathbf{A} = \mathbf{L} + \mathbf{Q}) \tag{1.7}$$

2. Theorem on the equivalence of the mean square stability and the stability of the trivial solutions of systems (1.1) and (1.7). Let $M_1, \ldots, M_N$ be arbitrary positive-definite matrices. We shall show that the initial distribution $\varphi(H, \{u_j\})$ can always be chosen such that $M_1(0), \ldots, M_N(0)$, where $M_k(0) = \{m_{ij}^k(0)\}$, turn out to equal $M_1, \ldots, M_N$ respectively. To do this we recall that if the probability density $p(y_1, \ldots, y_n)$ of an n-dimensional random variable $\xi = (\xi_1, \ldots, \xi_n)$ is Gaussian and equal to

$$p(y_1, \ldots, y_n) = \frac{1}{\sqrt{(2\pi)^n \det B}} \exp\left\{-\frac{1}{2}(B^{-1}y, y)\right\} \tag{2.1}$$

where B is an arbitrary positive-definite matrix, then B is the covariance matrix of the variable ξ, i.e., the elements b_{ij} of matrix B are the means: $b_{ij} = M(\xi_i\xi_j)$. Now let

$$\varphi(H, \{u_r\}) = \frac{1}{N}\frac{1}{\sqrt{(2\pi)^n N^n \det M_r}} \int_H \exp\left\{-\frac{1}{2}\left(\frac{1}{N}M_r^{-1}x, x\right)\right\} dx \tag{2.2}$$

Then it is clear that $M_r(0) = M_r$.

Theorem 1. For the trivial solution of system (1.1) to be asymptotically mean square stable, it is necessary and sufficient that the trivial solution of system (1.5) (or, (1.4), (1.7)) be asymptotically stable.

Proof. Sufficiency follows from the equality

$$M(\|x(t; x^\circ, u_k)\|^2) = \sum_{r=1}^{N}\sum_{i=1}^{n} m_{ii}^r(t)$$

where the $m_{ii}^r(t)$ are the corresponding coordinates of the solution of system (1.4), satisfying the initial data

$$m_{ij}^r(0) = \begin{cases} x_i^\circ x_j^\circ, & r = k \\ 0, & r \neq k \end{cases}$$

i.e., the initial data which correspond to the distribution concentrated at the point (x°, u_k).

Necessity. Let the trivial solution of system (1.1) be asymptotically mean square stable. This signifies that the mean

$$T_t f_{ii}^r(x, u_s) = \int_X \sum_{k=1}^{N} f_{ii}^r(y, u_k) P(t, x, u_s, dy, \{u_k\}) \tag{2.3}$$

tends to zero as $t \to \infty$ for all i and r. In formula (2.3), X is the n-dimensional space of the variables $x_1, \ldots, x_n$, while $P(t, x, u_s, H, \{u_k\})$ is the Markov transition function of the process $\{x(t), u(t), 0 \leqslant t < \infty\}$ (see the notation in [2, 4]). It is not difficult to obtain

$$\lim_{t\to\infty} T_t f_{ij}^r(x, u_s) = 0 \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n;\ r = 1, \ldots, N) \tag{2.4}$$

Let $\varphi(H, \{u_i\})$ satisfy (2.2). The solution of system (1.5), $M_r(t) = \{m_{ij}^r(t)\}$, satisfying the initial condition $M_r(0) = M_r$, can be written as

$$m_{ij}^r(t) = (T_t f_{ij}^r(x, u_s), \varphi(H, \{u_s\})) = \int_X \sum_{s=1}^{N} T_t f_{ij}^r(x, u_s)\, \varphi(dx, \{u_s\}) \tag{2.5}$$

By virtue of (2.4), $m_{ij}^r(t)$ tends to zero as $t \to \infty$. Thus, any solution of system (1.5), corresponding to the initial condition $M_r(0) = M_r$, where M_r $(r = 1, \ldots, N)$ is an arbitrary positive-definite matrix, tends to zero as $t \to \infty$. Since an arbitrary symmetric matrix can be represented as the difference between two positive-definite matrices, it follows that any solution of system (1.5) tends to zero as $t \to \infty$, i.e., the trivial solution of this system is asymptotically stable. The theorem is proved.

We note that for the sake of brevity in what follows, together with the expressions: asymptotic mean square stability of the trivial solution of system (1.1) or the asymptotic stability of the trivial solution of system (1.5), we shall also use the expressions: stability of system (1.1), stability of system (1.5), stability of operator A, etc.

3. The positiveness of solutions. We consider a finite-dimensional normed spaced $\mathbf{M}^N$ of vectors $M = (M_1, \ldots, M_N)$, whose components are symmetric $n \times n$ matrices. In what follows, as a rule we do not use an actual form of the norm for the vector M and, therefore, by the notation $\| M \|$ we mean some definite norm. We introduce the class $\mathbf{K}^1$ of nth - order nonnegative definite matrices. We shall write $P_1 \geqslant R_1$ if $P_1 - R_1 \in \mathbf{K}^1$, i.e., if $P_1 - R_1$ is a nonnegative-definite matrix, and $P_1 > R_1$ if $P_1 - R_1$ is a positive-definite matrix. We see that $\mathbf{K}^1$ is a cone in $\mathbf{M}^1$ [5], that this cone is reproducing, i.e., any element $M_1 \in \mathbf{M}^1$ is representable as the difference of two elements of $\mathbf{K}^1$, and that the interior of this cone consists of positive-definite matrices.

We consider a cone $\mathbf{K}^N \subset \mathbf{M}^N$ consisting of vectors M whose components are nonnegative-definite matrices. Let $P \in \mathbf{M}^N$, $R \in \mathbf{M}^N$, $P = (P_1, \ldots, P_N)$, $R = (R_1, \ldots, R_N)$. We write $P \geqslant R$ if simultaneously $P_1 \geqslant R_1, \ldots, P_N \geqslant R_N$, and $P > R$ if simultaneously $P_1 > R_1, \ldots, P_N > R_N$. The notation $P \geqslant R$ is equivalent to the inclusion $P - R \in \mathbf{K}^N$. The cone $\mathbf{K}^N$ is reproducing. A linear operator $\mathbf{B}$ on $\mathbf{M}^N$ is called nonnegative if it maps the cone $\mathbf{K}^N$ into itself: $\mathbf{B}\mathbf{K}^N \subset \mathbf{K}^N$, and $\mathbf{B}$ is called positive if $\mathbf{B}M > 0$ is fulfilled for any $M > 0$. Any nonnegative operator $\mathbf{B}$ possesses the property of monotonicity, i.e., $\mathbf{B}P \geqslant \mathbf{B}R$ follows from $P \geqslant R$. Let us write the solution of system (1.7), satisfying the initial condition $M(0) = M^\circ$, in the form

$$M = e^{\mathbf{A}t} M^\circ, \qquad e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \ldots + \frac{1}{n!}\mathbf{A}^n t^n + \ldots \tag{3.1}$$

where $\mathbf{I}$ is the identity operator in the space $\mathbf{M}^N$.

Lemma 1. The operator $e^{\mathbf{A}t}$ is nonnegative for any $t \geqslant 0$.

Proof. Let $M^\circ = (M_1^\circ, \ldots, M_N^\circ) \geqslant 0$, i.e., $M_1^\circ \geqslant 0, \ldots, M_N^\circ \geqslant 0$. We select the distribution $\varphi(H, \{u_r\})$ such that $m_{ij}^r(0) = (f_{ij}^r, \varphi(H, \{u_s\}))$ and $M_r^\circ = \{m_{ij}^r(0)\}$. It was shown above that by some means we can select such a distribution in the case of the positive-definite matrices $\overset{\circ}{M}_1, \ldots, \overset{\circ}{M}_N$. A similar distribution can be found also for nonnegative-definite matrices. Furthermore, because the operator $e^{\mathrm{A}t}$ is continuous and the vectors $M > 0$ comprise the interior of the cone K^N, it is sufficient to prove the nonnegativeness of operator $e^{\mathrm{A}t}$ for the vectors $M > 0$.

We consider the quadratic form

$$(F^r a, a) = \sum_{i, j=1}^{n} f_{ij}^r a_i a_j$$

where $a = (a_1, \ldots, a_n)$ is a vector with real coordinates. The matrix of this quadratic form is $\{x_i x_j\}$ for $u = u_r$ and equals zero for $u \neq u_r$, therefore this form takes a nonnegative value $(F^r a, a) \geqslant 0$ for any vectors $x = (x_1, \ldots, x_n)$, $a = (a_1, \ldots, a_n)$ and any u_s. Hence $(T_t (F^r a, a), \varphi) \geqslant 0$ since $(T_t (F^r a, a), \varphi)$ is the mean of function $(F^r a, a)$ at the instant t under the initial distribution $\varphi (H, \{u_s\})$. Then

$$(T_t (F^r a, a), \varphi) = \left(T_t \sum_{i, j=1}^{n} f_{ij}{}^r a_i a_j, \varphi\right) = \sum_{i, j=1}^{n} m_{ij}{}^r (t)\, a_i a_j \geqslant 0$$

The latter inequality is valid for all a and, therefore, the quadratic form $(M_r (t)\, a, a)$ is nonnegative definite, i.e., $M_r (t) \geqslant 0$ and $M (t) = e^{\mathrm{A}t} M^\circ \geqslant 0$. The lemma is proved.

We note that when Eq. (1.1) is scalar the operator A is a numerical matrix with nonnegative off-diagonal elements. The nonnegativeness of the solutions for differential equations with such matrices is well known ([6], p. 207).

Theorem 2. The operator A has a real eigenvalue which is not less than the real part of any of the remaining eigenvalues. At least one nonnegative-definite eigenvector corresponds to this eigenvalue.

Proof. According to Lemma 1 and the Frobenius theorem [5] the operator $e^{\mathrm{A}t_0}$, where $t_0 > 0$ is arbitrary, possesses a positive eigenvalue μ_0 such that all the remaining eigenvalues do not exceed μ_0 in absolute value. The vector $M^\circ \geqslant 0$ corresponds to this eigenvalue μ_0 i.e.,

$$e^{\mathrm{A}t_0} M^\circ = \mu_0 M^\circ \tag{3.2}$$

Suppose that all the eigenvalues of operator A are distinct. It is well known that if λ is an eigenvalue of operator A, then $e^{\lambda t_0}$ is an eigenvalue of operator $e^{\mathrm{A}t_0}$. We can choose t_0 such that all the eigenvalues of operator $e^{\mathrm{A}t_0}$ also are distinct and that the realness of the eigenvalue λ for operator A follows from the realness of the eigenvalue $e^{\lambda t_0}$ for operator $e^{\mathrm{A}t_0}$. We shall take it that the t_0 in (3.2) has been chosen in just this way. Then $\lambda_0 = t_0^{-1} \ln \mu_0$ is an eigenvalue of operator A, and the real parts of all the remaining eigenvalues do not exceed λ_0. Further, for such t_0 all the eigenvectors of operators $e^{\mathrm{A}t_0}$ and A coincide. Therefore, $\mathrm{A}M^\circ = \lambda_0 M^\circ$ and, consequently, we have proven the theorem for the case of distinct eigenvalues of operator A. The theorem's proof in the general case is obtained from the preceding by calling on the continuous properties of eigenvalues and eigenvectors.

4. Necessary and sufficient stability conditions. 1°. Theorem 1 reduces the problem of asymptotic mean square stability of the trivial solution of system (1.1) to the usual asymptotic stability of the trivial solution of system (1.7). System (1.7) is of order $\frac{1}{2} n (n + 1) N$ and we apply the usual methods for investigating it. In particular, if we apply the Liapunov function method, then for the stability investigation we require quadratic forms of that same order, while to look for a quadratic form we need, consequently, to find its parameters from a system of linear equations in number $\frac{1}{2}(\frac{1}{2}n(n+1)N+1)\frac{1}{2}n(n+1)N$. However, for system (1.7), because the operator $e^{\mathbf{A}t}$ is positive, the stability problem can be reduced to looking for only $\frac{1}{2}n(n+1)N$ parameters in all. Below we derive the necessary and sufficient stability conditions, obtained previously by other means in [1], by making use of the idea of the positiveness of the solutions of Eqs. (1.7). We prove a lemma as a preliminary.

Lemma 2. Let the operator $\mathbf{B}$ in the equation

$$dM/dt = \mathbf{B}M \tag{4.1}$$

be such that the operator $\mathbf{B}+\rho\mathbf{I}$ is nonnegative for some $\rho > 0$. Then, for the stability of operator $\mathbf{B}$ it is sufficient that the equation

$$\mathbf{B}M = C \tag{4.2}$$

possesses a solution $M > 0$ ($M < 0$) for some vector $C < 0$ ($C > 0$) and it is necessary that Eq.(4.2) possesses a solution $M > 0$ ($M < 0$) for any $C < 0$ ($C > 0$).

Proof. Sufficiency. Let equality (4.2) be fulfilled for some $M > 0$ and $C < 0$. Then when $C < \lambda M$ the inequality $\mathbf{B}M \leqslant \lambda M$ holds for some $\lambda < 0$. Hence $(\mathbf{B}+\rho\mathbf{I})M \leqslant (\lambda+\rho)M$. Since $\mathbf{B}+\rho\mathbf{I}$ is a nonnegative operator, for any k

$$(\mathbf{B}+\rho\mathbf{I})^k M \leqslant (\lambda+\rho)^k M$$

Hence

$$e^{(\mathbf{B}+\rho\mathbf{I})t} M \leqslant \sum_{k=0}^{\infty} \frac{(\lambda+\rho)^k t^k}{k!} M = e^{(\lambda+\rho)t} M, \qquad e^{\mathbf{B}t} M \leqslant e^{\lambda t} M$$

Let us now show that under the lemma's hypotheses the operator $e^{\mathbf{B}t}$ is nonnegative. Indeed, if $M \geqslant 0$, then $(\mathbf{B}+\rho\mathbf{I})^k M \geqslant 0$, whence $e^{(\mathbf{B}+\rho\mathbf{I})t} M \geqslant 0$, which is equivalent to the inequality $e^{\mathbf{B}t} M \geqslant 0$. Because M is positive definite, for any $\overline{M} \geqslant 0$ we can find a number $\nu > 0$ such that $\overline{M} \leqslant \nu M$, therefore,

$$0 \leqslant e^{\mathbf{B}t}\overline{M} \leqslant \nu e^{\mathbf{B}t} M \leqslant \nu e^{\lambda t} M$$

Consequently, for any $\overline{M} \geqslant 0$ the solution $e^{\mathbf{B}t}\overline{M}$ of Eq. (4.1) tends to zero as $t \to \infty$ (recall that $\lambda < 0$). But from this it follows that $e^{\mathbf{B}t}M \to 0$ as $t \to \infty$ for any $M \in \mathrm{M}^N$, since the cone $\mathbf{K}^N$ is reproducing. We have proven the sufficiency.

Necessity. The trivial solution of Eq. (4.1) is asymptotically stable, therefore, all the eigenvalues of operator $\mathbf{B}$ lie in the left halfplane and operator $\mathbf{B}$ has an inverse $\mathbf{B}^{-1}$ for which the formula

$$\mathbf{B}^{-1}C = -\int_0^\infty e^{\mathbf{B}t}\, C\, dt \tag{4.3}$$

is valid. It follows from formula (4.3) that the operator $-\mathbf{B}^{-1}$ is positive. This is so because the operator $e^{\mathbf{B}t}$ is nonnegative for all $t \geqslant 0$ and because $e^{\mathbf{B}t}C > 0$ if $C > 0$ for all sufficiently small t. Writing the solution of Eq. (4.2) in the form $M = \mathbf{B}^{-1}C$, we conclude the proof of the necessity and, along with this, of the lemma.

Theorem 3 [1]. For the asymptotic stability of the trivial solution of Eq. (1.7) it is necessary that for any $C < 0$ ($C > 0$) the solution of the equation

$$\Lambda M = C \tag{4.4}$$

exists and is positive definite $M > 0$ ($M < 0$), and it is sufficient that for some $C < 0$ ($C > 0$) the solution M of Eq.(4.4) satisfies the condition $M > 0$ ($M < 0$).

Proof. Necessity is proved in the same way as the necessity in Lemma 2.

Indeed, according to Lemma 1 the operator $e^{\mathrm{A}t}$ is nonnegative, and, moreover, for small t, $e^{\mathrm{A}t}\, C < 0$ follows from the condition $C < 0$. Hence, the solution of Eq. (4.4) $M = \mathrm{A}^{-1}C > 0$.

Sufficiency. Suppose that $\overline{M} > 0$ is a solution of Eq. (4.4) for some $\overline{C} < 0$. Let us write Eq. (4.4) out in more detail

$$\begin{aligned} L_1M_1 + q_{21}M_2 + \dots + q_{N1}M_N &= C_1 \\ q_{12}M_1 + L_2M_2 + \dots + q_{N2}M_N &= C_2 \\ \dots\dots\dots\dots\dots\dots\dots \\ q_{1N}M_1 + q_{2N}M_2 + \dots + L_NM_N &= C_N \end{aligned} \tag{4.5}$$

For $M = \overline{M} = (\overline{M}_1, \dots, \overline{M}_N)$ the kth equation of system (4.5) can be rewritten as

$$L_k\overline{M}_k = C_k - q_{1k}\overline{M}_1 - \dots - q_{k-1k}\overline{M}_{k-1} - q_{k+1k}\overline{M}_{k+1} - \dots - q_{Nk}\overline{M}_N \tag{4.6}$$

Since the right-hand side of this equality is a negative-definite matrix, while $M_k > 0$, we have that the operator L_k is stable by a well known theorem of Liapunov. In other words, the eigenvalues of each matrix $A_k - {}^1/_2\, q_kE$ $(k = 1, \dots, N)$ lie in the left half-plane. Hence it follows that operators L_k have inverses L_k^{-1}. We note in passing that the operators $-L_k^{-1}$ are positive in the space M^1 with cone K^1. Multiplying the kth equation of (4.5) on the left by the operator $-L_k^{-1}$ we obtain

$$\begin{aligned} -M_1 - q_{21}L_1^{-1}M_2 - \dots - q_{N1}L_1^{-1}M_N &= -L_1^{-1}C_1 \\ -q_{12}L_2^{-1}M_1 - M_2 - \dots - q_{N2}L_2^{-1}M_N &= -L_2^{-1}C_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -q_{1N}L_N^{-1}M_1 - q_{2N}L_N^{-1}M_2 - \dots - M_N &= -L_N^{-1}C_N \end{aligned} \tag{4.7}$$

We denote $\mathrm{B} = -\mathrm{L}^{-1}\mathrm{A} = -\mathrm{I} - \mathrm{L}^{-1}\mathrm{Q}$. We see that the operator $\mathrm{B} + \mathrm{I}$ is nonnegative and the equality $\mathrm{B}\overline{\mathrm{M}} = -\mathrm{L}^{-1}\overline{C}$ holds. Since $-\mathrm{L}^{-1}\overline{C} < 0$, by virtue of Lemma 2 we obtain that operator B is stable. Hence, once again from Lemma 2, we conclude that for any $C < 0$ the solution M of the equation

$$\mathrm{B}M \equiv -\mathrm{L}^{-1}\mathrm{A}M = -\mathrm{L}^{-1}C$$

is positive definite, $M > 0$. Therefore, for any $C < 0$ the solution M of the equation $\mathrm{A}M = C$ is positive definite. Hence for any $C \leqslant 0$ the solution M of the equation $\mathrm{A}M = C$ is nonnegative definite, $M \geqslant 0$. Therefore, from the equality $\mathrm{A}M = \lambda M$ it follows, when $M \geqslant 0$ that $\lambda \leqslant 0$. According to Theorem 2 the operator A has a real eigenvalue λ_0 to which corresponds the eigenvector $M^\circ \geqslant 0$ and, moreover, the real parts of all the remaining eigenvalues do not exceed λ_0. Consequently, to prove the stability of operator A it is sufficient to prove that $\lambda_0 < 0$. Since $\mathrm{A}M^\circ = \lambda_0 M^\circ$ and $M^\circ \geqslant 0$, we have $\lambda_0 \leqslant 0$. But $\lambda_0 \neq 0$ because the operator A is invertible by virtue of the invertibility of operators B and L. Thus, $\lambda_0 < 0$ and operator A is stable. The theorem is proved.

Corollary 1. The stability of the matrices $A_k - {}^1/_2 q_k$ $(k = 1, \dots, N)$ and the positiveness of the operators $-L_k^{-1}$ are the necessary conditions for the asymptotic mean square stability of the trivial solution of system (1.1).

This assertion follows clearly from the sufficiency proof. A generalization of this corollary is given below.

Corollary 2. We consider a subsystem of system (1.7), which is defined by the set of indices $i_1 < i_2 < \dots < i_k$ (or, equivalently, by one of the principal minors of matrix A) and has the form

$$\frac{dM_{i_1}}{dt} = L_{i_1}M_{i_1} + q_{i_2 i_1}M_{i_2} + \ldots + q_{i_k i_1}M_{i_k} \tag{4.8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dM_{i_k}}{dt} = q_{i_1 i_k}M_{i_1} + q_{i_2 i_k}M_{i_2} + \ldots + L_{i_k}M_{i_k}$$

If the trivial solution of system (1.7) is asymptotically stable, then the trivial solution of system (4.8) also is asymptotically stable. We omit the proof of this corollary.

Corollary 3. For the stability of operator A it is necessary and sufficient that the operator $-\mathrm{L}^{-1}\mathrm{A}$ be stable.

This follows from the sufficiency proof in Theorem 3.

A note on linear Liapunov functions. We consider the case when Eq. (1.1) is scalar

$$dx/dt = a\,(u)\,x \tag{4.9}$$

The system of Eqs. (1.5) has the form

$$\frac{dM_r}{dt} = (2a_r - q_r)\,M_r + \sum_{i \neq r} q_{ir}M_i \tag{4.10}$$

where M_r $(r = 1, \ldots, N)$ are numbers, $a_r = a\,(u_r)$. The matrix of system (4.10) is a numerical matrix A. We can show that any matrix A with nonnegative off-diagonal elements can be related with an equation of form (4.9) as the matrix of the corresponding system for the moments. Therefore, by virtue of Theorem 3 the operator A^* is stable if and only if the equation $\mathrm{A}^*B = C$, where $C < 0$ (i.e., all the numbers $C_1, \ldots, C_N$ are negative), has a solution $B > 0$. For system (4.10) we consider the function

$$V\,(M_1, \ldots, M_N) = (B,\ M) = \sum_{k=1}^{N} B_k M_k$$

This function is positive in cone K^N of vectors with nonnegative coordinates (if, it is clear, $M \neq 0$). The total derivative of this function relative to system (4.10) is

$$\frac{dV}{dt} = \left(B,\ \frac{dM}{dt}\right) = (B,\ \mathrm{A}M) = (\mathrm{A}^*B,\ M) = (C,\ M) \tag{4.11}$$

i.e., is negative in cone K^N. Thus we get that for the asymptotic stability of the trivial solution of system (4.10) it is necessary and sufficient that there exist a linear function which is positive in K^N and is such that its derivative relative to the system is a linear function negative in K^N. It is natural to call such functions linear Liapunov functions. We confine ourselves to this note and do not consider here the general case from the viewpoint of linear Liapunov functions.

2°. A simple stability criterion is known [7, 8] for matrices with nonnegative off-diagonal elements. The application of this criterion to the matrix A of system (4.10) leads to the proposition: for the stability of matrix A it is necessary and sufficient to fulfill the N inequalities

$$2a_1 - q_1 < 0, \quad \begin{vmatrix} 2a_1 - q_1 & q_{21} \\ q_{12} & 2a_2 - q_2 \end{vmatrix} > 0, \ldots, \quad (-1)^N \begin{vmatrix} 2a_1 - q_1 & q_{21} \ldots q_{N1} \\ q_{12} & 2a_2 - q_2 \ldots q_{N2} \\ \cdots & \cdots \\ q_{1N} & q_{2N} \ldots 2a_N - q_N \end{vmatrix} > 0$$

A generalization of this result to the case when the matrices $A_1, \ldots, A_N$ commute is given in Theorem 4.

Lemma 3. Suppose that the matrices $A_1, \ldots, A_N$ commute. Then, all the operators $L_1, \ldots, L_N$, all possible linear combinations of their products, and the inverses of these combinations (if these inverses exist) also commute.

Proof. The commutability of the operators $L_1, \ldots, L_N$ is verified directly. The rest follows from the general results concerning linear operators in a finite-dimensional space.

For the statement of the next theorem we introduce the operators

$$\Lambda_1 = L_1,\ \Lambda_2 = \begin{vmatrix} L_1 & q_{21} \\ q_{12} & L_2 \end{vmatrix}, \ldots,\ \Lambda_N = \begin{vmatrix} L_1 & q_{21} \ldots q_{N1} \\ q_{12} & L_2 \ldots q_{N2} \\ \cdots & \cdots \\ q_{1N} & q_{2N} \ldots L_N \end{vmatrix}$$

where each operator Λ_k $(k = 1, \ldots, N)$ is obtained by a formal development of the determinant.

Theorem 4. If the matrices $A_1, \ldots, A_N$ commute, then for the stability of operator A it is necessary that the operators $(-1)^k \Lambda_k^{-1}$ be positive and it is sufficient that for any k the equation

$$(-1)^k \Lambda_k M_k = C_k \tag{4.12}$$

have a positive-definite solution $M_k > 0$ for some C_k.

Without carrying out a detailed proof of the theorem we note that it is based on the application of the Gaussian elimination method to system (4.5). The operators encountered during the computations commute by virtue of Lemma 3, therefore, all the calculations valid for the scalar case [7] carry over without any alterations.

Note. Theorem 4 is valid for $N = 2$ in the general (and not only in the commutative) case.

We note also the necessary and sufficient conditions using the spectral properties of the operator $-\mathrm{L}^{-1}\mathrm{Q}$. In Corollary 3 to Theorem 3 we noted that the stability of operator A is equivalent to the stability of the operator $-\mathrm{L}^{-1}\mathrm{A} = -\mathrm{I} - \mathrm{L}^{-1}\mathrm{Q}$. But the operator $-\mathrm{L}^{-1}\mathrm{Q}$ is nonnegative. Consequently, by the Frobenius theorem this operator has a nonnegative eigenvalue μ_0 such that all the remaining eigenvalues do not exceed μ_0 in absolute value. We see that for the stability of operator $-\mathrm{L}^{-1}\mathrm{A}$, and, consequently, also A, it is necessary and sufficient that $\mu_0 < 1$.

Theorem 5. For the stability of operator A it is necessary that

$$\lim_{k\to\infty} \| (-\mathrm{L}^{-1}\mathrm{Q})^k \| = 0 \tag{4.13}$$

and it is sufficient that for some $M > 0$

$$\lim_{k\to\infty} (-\mathrm{L}^{-1}\mathrm{Q})^k M = 0 \tag{4.14}$$

Proof. Necessity. If A is stable, then $\mu_0 < 1$. But, by the Frobenius theorem, $\rho(-\mathrm{L}^{-1}\mathrm{Q}) = \mu_0$, where $\rho(-\mathrm{L}^{-1}\mathrm{Q})$ is the spectral radius of the operator $-\mathrm{L}^{-1}\mathrm{Q}$. We apply I. M. Gel'fand's formula to compute the spectral radius

$$\lim_{k\to\infty} \sqrt[k]{\| (-\mathrm{L}^{-1}\mathrm{Q})^k \|} = \rho(-\mathrm{L}^{-1}\mathrm{Q}) \tag{4.15}$$

Equation (4.13) follows from (4.15) because $\rho(-\mathrm{L}^{-1}\mathrm{Q}) < 1$.

Sufficiency. If $M > 0$, then for a sufficiently large k from (4.14) we have the inequality

$$(-\mathrm{L}^{-1}\mathrm{Q})^k M < \alpha M \quad (\alpha < 1) \tag{4.16}$$

The operator $(-L^{-1}Q)^k$ is nonnegative, therefore, from (4.16) we obtain that its eigenvalue largest in absolute value, which equals μ_0^k, is not greater than α. Therefore, $\mu_0 < 1$, and the theorem is proved.

The noted spectral property of operator $-L^{-1}Q$ and Theorem 5 permit us to make use of the highly developed theory of positive operators [5, 9, 10] to establish the stability or instability of operator A.

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